# Using unitary operations to preserve quantum states in the presence of relaxation

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(Dated: January 29, 2003)

When a quantum system interacts with an external environment, it undergoes the loss of quantum correlation (decoherence) and the loss of energy (relaxation) and eventually all of the quantum information becomes classical. Here we show a general principle to use unitary operations to establish and preserve particular non-equilibrium states in arbitrary relaxing quantum systems. We elucidate these concepts with examples of state preservation in one-spin and two-spin entangled systems.

PACS numbers: 03.65.Yz,03.67.-a,33.25.+k,82.56.Na

Quantum systems undergo damping and decoherence when they interact with thermodynamic systems. Preventing such non-unitary behavior is one of the biggest challenges facing the engineering of quantum information technology.

Much work has been done to develop quantum error correction schemes to correct these non-unitary errors [1] [2]. When the errors are below known thresholds [3], these schemes can preserve quantum information for an arbitrarily long time. On the other hand, error correction is likely to be difficult because the required fidelity is beyond current practice and a large supply of ancilla qubits is needed [4][5][6].

When the dissipation modes are purely decoherence processes, Decoherence Free Subspaces that are unaffected by the decoherence can be relabeled and used for storing and processing quantum information [7][8]. In this paper, we will look at the situation when the dissipation is damping or relaxation. Surprisingly in this case, unitary operations can be used to preserve a large submanifold of quantum states for an arbitrarily long time. We will outline the geometry of such situations and then detail two examples. The first example will illustrate the dynamics of one-spin systems. The second example will show how to preserve a pair of coupled spins in a particular entangled state.

First, consider a general open quantum system of dimension $N$ with density matrix $\rho$. A general time evolution of this system is given by the mapping

$$\rho \mapsto \mathcal{E}_t(\rho) = \sum_k E(t)_k \rho E(t)_k^\dagger \tag{1}$$

where the $E_k$ satisfy $\sum_k E_k(t)^\dagger E_k(t) = \mathbb{1}$ for all $t$ and are called Kraus operators [9]. If our open system evolution is Markovian, and satisfies the algebraic property $\mathcal{E}_t\mathcal{E}_s = \mathcal{E}_{t+s}$, then the evolution is called a quantum dynamical semigroup. The dynamics are completely determined by the generator of this semigroup. The most general differential equation for such a generator is given by the Lindblad Equation

$$\dot{\rho} = -i[H,\rho] + \frac{1}{2}\sum_{k=0}^{K}[L_k, \rho L_k^\dagger] + [L_k\rho, L_k^\dagger] \tag{2}$$

where $H$ is a Hermitian matrix and $L_k$ are a set of $N \times N$ matrices [10]. If the $L_k$ are identically zero, then the Lindblad equation reduces to a Schrödinger equation and we call the matrix $H$ the Hamiltonian part of the dynamics. In turn, the set terms of the Lindblad equation involving the operators $L_k$ is called the dissipative part of the Lindblad equation.

Denote the trace zero Hermitian matrices $\mathfrak{su}(N)$ and choose a basis, $F_k$, satisfying the orthogonality conditions $\mathrm{Tr}(F_jF_k) = \delta_{jk}$. Every density matrix $\rho$ can be written as a sum

$$\rho = \frac{\mathbb{1} + \sum_{k=1}^{N^2-1} r_k F_k}{N} \tag{3}$$

If we change our representation and define the coherence vector, $\vec{r} = (r_k)$, then equation 2 can be written as a standard form ordinary differential equation

$$\dot{\vec{r}} = A\vec{r} + B\vec{r} + \vec{c} \tag{4}$$

Here $A$ corresponds to the Hamiltonian part of the Lindblad equation while $B$ and $\vec{c}$ correspond to the dissipative part [11].

We will focus on a special type of quantum dynamical semigroups called relaxing semigroups. A semigroup is relaxing if for any initial state $\vec{r}(0)$,

$$\lim_{t\to\infty} \vec{r}(t) = \vec{r}_f. \tag{5}$$

Such a situation occurs whenever the matrix $A + B$ is invertible and the vector $\vec{c}$ is also nonzero. By setting $d\vec{r}/dt = 0$, we see that $\vec{r}_f = -(A+B)^{-1}\vec{c}$ is the unique fixed point of the evolution. If the Lindblad equation has a unique fixed point, $\vec{r}_f$, then this fixed point is a global attractor. Indeed, in matrix form we find that the evolution is given in coherence vector form by

$$\vec{r}(t) = e^{(A+B)t}(\vec{r}_0 - \vec{r}_f) + \vec{r}_f \tag{6}$$

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where $\vec{r}_f$ is the global fixed point.

Relaxing semigroups occur frequently in physical systems where the equilibrium state of a system is known *a priori*. For example, in a liquid state nuclear magnetic resonance experiment, the system will always return to an equilibrium Boltzmann distribution which is solely a function of the applied magnetic field and the temperature. In what follows, we will show that when a quantum system evolves as a relaxing semigroup, unitary controllers can act to stabilize a variety of known states.

Let us restrict attention to the following control scenario. Suppose a quantum system evolves as a relaxing semigroup, but that we can apply an arbitrary controlling Hamiltonian, $H_c$, to the system, but that we cannot adjust the dissipative terms. In the coherence vector representation we have

$$\dot{\vec{r}} = A_c\vec{r} + B\vec{r} + \vec{c} \tag{7}$$

The Hamiltonian control cannot prevent relaxation, as the eigenvalues of the matrix $A_c + B$ will still have negative real parts, but the following proposition shows that the controller shifts the fixed point of the relaxing semigroup.

**Proposition 1** *Let $B$ be the dissipative part of the Lindblad equation for a relaxing semigroup. If $B$ is diagonalizable then for any Hamiltonian part $A$, the matrix $A+B$ is invertible.*

**Proof** Let $\langle\ ,\ \rangle$ be an inner product on $\mathbb{R}^N$. Then the skew symmetry of $A$ implies that $\langle\vec{r}, A\vec{r}\rangle = 0$ for all $\vec{r}$. There exists a basis $f_j$ for $B$ such that

$$\langle\vec{r}, B\vec{r}\rangle = \sum_{k=1}^{N}(\lambda_k + i\omega_k)\langle r, f_k\rangle^2 \tag{8}$$

where the $\lambda_k < 0$. The real part of the inner product is negative for any nonzero $\vec{r}$. This in turn means that

$$\mathrm{Re}(\langle\vec{r}, (A+B)\vec{r}\rangle) < 0 \tag{9}$$

for all $\vec{r} \neq 0$ which completes the proof. ■

If we apply a Hamiltonian $A_c$ then the state $\vec{r}_f = (A_c + B)^{-1}\vec{c}$ becomes the global attracting fixed point of our quantum system and $A_c$ is a stabilizing controller on our system. In particular, this means that when $A_c$ is applied, the system's steady state is $\vec{r}_f$, and and the system will flow to $\vec{r}_f$ independent of the initial state. Hence the set

$$\mathcal{C} = \{\vec{r} = -(A+B)^{-1}\vec{c}\,|\,A \text{ is a Hamiltonian}\} \tag{10}$$

can be made into fixed points of a relaxing semigroup using control Hamiltonians. Since these states will be stabilized by the dynamics, we will refer to $\mathcal{C}$ as the set of stabilizable states of our semigroup.

We must note that the stabilizable states will in general be mixed states as the length of the vector $\vec{r}_f$ will vary with the applied Hamiltonian. However, we will see that they can be useful for monitoring quantum systems and for preserving entanglement. The following theorem describes the geometry of the set $\mathcal{C}$. In the proof, we will switch between the density matrix and coherence vector representations.

**Theorem 2** *If the fixed point, $\rho_{eq}$, of a relaxing semigroup has non-degenerate eigenvalues, then the set of stabilizable states is a simply connected $N^2 - N$ manifold containing the fixed point of the process and having the maximally mixed state in its closure.*

**Proof** Let $\rho_{eq}$ be the fixed point of the quantum process with corresponding coherence vector $\vec{r}_{eq}$. Consider a small perturbation $\vec{r} = \vec{r}_{eq} + \delta\vec{r}$. It is immediate to show that $A\vec{r} + \vec{b} = A\delta\vec{r}$.

For an infinitesimal time $\Delta t$, we have that $\rho(\Delta t) = \rho + A\delta\vec{r}\Delta t$. If over this time, the eigenvalues of $\rho(\Delta t)$ are the same as those of $\rho(0)$, then there exists a unitary operator $U$ with $U\rho(t)U^\dagger = \rho(0)$ and hence $\rho$ is the fixed point of the process

$$\ldots\exp(-iH_c\Delta t)\mathcal{E}_t\exp(-iH_c\Delta t)\mathcal{E}_t\ldots \tag{11}$$

which is generated by the Lindblad equation with $H = H_c$.

Let $|\psi_n\rangle$ be an orthonormal eigenbasis for $\rho_{eq}$ with corresponding eigenvalues $p_1 > \ldots > p_N$. We want to show that there is an $N^2 - N$ dimensional neighborhood of $\rho_{eq}$ where the eigenvalues are unchanged under such a small perturbation. Since $\rho$ and $\mathcal{E}_{\Delta t}(\rho)$ are perturbations of $\rho_{eq}$, we can calculate the change in the eigenvalues

$$\begin{aligned}\Delta p_n &= \langle\psi_n|\rho_{eq} + \delta\rho - \mathcal{E}_{\Delta t}(\rho_{eq} + \delta\rho)|\psi_n\rangle \\ &= \langle\psi_n|\frac{A\delta\vec{r}}{N}|\psi_n\rangle\end{aligned} . \tag{12}$$

The set of matrices $M \in \mathfrak{su}(N)$ such that $\langle\psi_n|M|\psi_n\rangle = 0$ has dimension $N^2 - N$ as it corresponds to those traceless Hermitian matrices with zeros on the diagonal.

If $\rho$ is in the set of stabilizable states and has a corresponding Hamiltonian $H$, then there is a corresponding $\rho_\mu$ for $\mu H$. At the limit where $\mu = \infty$, $\rho_\infty = \mathbb{1}/N$. Hence, $\mathbb{1}/N$ is a limit point of $\mathcal{C}$. ■

The preceding argument sets an upper bound on the dimensionality of the space of stabilizable states. If the fixed point has degenerate eigenvalues or the set of controller Hamiltonians is restricted to a subspace of $\mathfrak{su}(N)$ then $\mathcal{C}$ will have smaller dimension.

The utility of this formalism can be explored in a one-spin example. Consider the process of damping to the $Z$ eigenket $|\uparrow\rangle$. In terms of the Bloch vector, the system will relax with time constant $\gamma_1 = 1/T_1$ along the $Z$-axis and decohere with time constant $\gamma_2 = 1/T_2 \geq \gamma_1/2$ in the $x$-$y$ plane.

The Lindblad equation which generates such a semigroup is given by equation 2 with $H = 0$, $K = 1$, and

$$L_0 = \sqrt{\gamma_1} I^+ \quad \text{and} \quad L_1 = \sqrt{\frac{\gamma_2}{2} - \frac{\gamma_1}{4}} Z \tag{13}$$

where $I^+ = (X + iY)/\sqrt{2}$ is the raising operator. In the form of equation 4 this amounts to the Bloch equations with $A = 0$,

$$B = \begin{pmatrix} -\gamma_2 & 0 & 0 \\ 0 & -\gamma_2 & 0 \\ 0 & 0 & -\gamma_1 \end{pmatrix}, \text{ and } \quad C = \begin{pmatrix} 0 \\ 0 \\ \gamma_1 \end{pmatrix} \tag{14}$$

Parametrizing the space of Hamiltonians in the Pauli basis gives the controller Hamiltonian

$$A = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}, \tag{15}$$

in terms of three parameter controls $\{u_x, u_y, u_z\}$ corresponding to rotations about the $x$,$y$, and $z$ axis respectively.

The fixed points of the Lindblad equation are given by the equation $(A + B)\vec{r} + \vec{c} = 0$ which can can be solved to find the manifold $\mathcal{C}$

$$\frac{1}{4} = (z - \frac{1}{2})^2 + \frac{\gamma_2}{\gamma_1}(x^2 + y^2) \tag{16}$$

$\mathcal{C}$ is an ellipsoid containing both the fixed point and the maximally mixed point as we proved earlier. Its minor axis is governed only by the ratio of $\gamma_1$ to $\gamma_2$.

We also find the appropriate open loop controllers to reach the state $(x, y, z)$ on $\mathcal{C}$ are

$$u_x = \tau_2 \frac{y}{z} \quad \text{and} \quad u_y = \tau_1 \frac{x}{z} \tag{17}$$

To ground this example in practice, let us describe how it is readily applied applied to pulsed NMR. We can asymptotically reach the desired steady state by applying the unitary operations $\exp(-i(u_x X + u_y Y)\Delta t)$ at a repetition rate of $\Delta t$. When $T_2$ (the transverse relaxation time $1/\gamma_2$) is comparable with $T_1$ (the longitudinal relaxation time $1/\gamma_1$), the steady state component of the Bloch vector in the $x$-$y$ plane can be asymptotically close to $\frac{1}{2}$. In the language of NMR, the steady state magnetization is equal to half of the peak magnetization from a $\pi/2$ pulse when $T_1 = T_2$. On the other hand, when $T_2 \ll T_1$, the steady-state magnetization approaches zero. Ernst and Anderson [12] and Freeman [13] derived these steady state from the Bloch equations, and our current formalism includes their results as a special case. Furthermore, by varying the pulse width, and in turn the steady state, we have experimentally demonstrated control over the NMR magnetization vector over times much larger than T1 as shown in figure 1.

Investigations into steady-state NMR on multiple spin systems has been less broadly investigated. It has proved

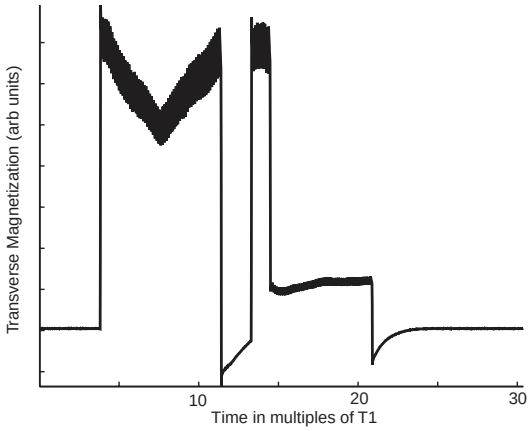


FIG. 1: Experimental data tracing out the letters "ML" in the transverse magnetization of an NMR spin system. We prepared a Copper Sulfate sample in water following the prescription in [14] to create time constants $T_1 = T_2 = 39 \pm 5$ms. The signal was measured using a Varian 500 MHz NMR spectrometer.

successful for the specific case of studying spin-lattice relaxation in dipolar solids [15][16], but a general theory for multiple spins has not been established. Our results readily extend to higher dimensional quantum systems, but parametrizing the set of stabilizable states and their corresponding controllers becomes much more difficult as the number of variables in the coherence vector scales quadratically with the number of levels. Nonetheless, we will demonstrate techniques for dealing with such larger systems and describe a particular example of using local controllers and an entangling operation to preserve a highly entangled state.

The entanglement of a pure state, $\rho = |\psi\rangle\langle\psi|$ of two spin half particles is defined as

$$E(\rho) = \mathrm{Tr}(\rho_1 \log \rho_1) = \mathrm{Tr}(\rho_2 \log \rho_2) \tag{18}$$

where $\rho_j$ denotes the partial trace over the Hilbert space of the $j$th spin. The quantity $E$ takes values between 0 and 1 and provides an information-like measure of the entanglement between a pair of spins. Correspondingly, an "ebit" is a unit of entanglement. One EPR pair has one ebit of entanglement. For a mixed state, we can define the entanglement of formation to be the minimum amount of entanglement required to create this mixed states from pure states [5]. Precisely,

$$E(\rho) = \min \sum_j p_j E(\psi_j) \tag{19}$$

where the minimum is taken over all combinations of $p_j$ and $|\psi_j\rangle$ which yield $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$. Wootters found a functional form for this quantity which involves extracting the eigenvalues of an algebraic function of the density matrix [17]. Using this metric, we show how to construct a stabilizable state with an entanglement 0.355.

To simplify the equations used to solve for the fixed point, we will restrict our attention to a simple model.

Consider a two-spin system where both spins undergo damping to the spin-up state identically and independently with constants $\gamma_1 = \gamma_2/2$. This corresponds to dissipative operators

$$L_0 = \sqrt{\gamma} I^+ \otimes \mathbb{1} \quad \text{and} \quad L_1 = \sqrt{\gamma} \mathbb{1} \otimes I^+ \tag{20}$$

Assume the spins are coupled via the Hamiltonian $H = JZ_1Z_2$. The fixed point of this evolution is the state $|\psi_0\rangle = |\uparrow\uparrow\rangle$.

Allow for only local Hamiltonians to be applied. Then the admissible Hamiltonians can parametrized as

$$H_c = u_{x_1}X_1 + u_{y_1}Y_1 + u_{z_1}Z_1 + u_{x_2}X_2 + u_{y_2}Y + u_{z_2}Z_2 \tag{21}$$

The coherence vector is given by $r_{jk} = \mathrm{Tr}(\sigma_j \otimes \sigma_k \rho)$ where $\sigma_j = \{\mathbb{1}, X, Y, Z\}$ for $j = 0, 1, 2, 3$. Putting this all together, we get the fifteen equations

$$A^{lm}_{jk} r_{lm} + B^{lm}_{jk} r_{lm} + C_{lm} = 0 \tag{22}$$

with the coefficients of A and B readily solved for by algebra.

Consider the Hamiltonian

$$H_c = \frac{4\sqrt{J}}{5} X_1 - JZ_1 + \frac{4\sqrt{J}}{5} X_2 - JZ_2 \tag{23}$$

and let $|\psi_1\rangle = |\uparrow\uparrow\rangle$ and $|\psi_2\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$. Inverting the system in equation 22 and taking the limit as $J$ approaches infinity, yields the fixed state

$$\rho_e = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2| \tag{24}$$

which indeed has the entanglement of formation of 0.355.

The rate at which the fixed point approaches infinity is plotted in figure 2. Even for relatively small ratios, $J/\gamma$, this state is close to $\rho_e$. In the context of quantum computation, this procedure could be used to make a "well" of entanglement. Spins that are coupled locally can be used to store a known entangled state and then their state can be swapped into another system which can process the entanglement for communication or computation.

We have shown a method for analyzing relaxing semigroups and have also shown that by applying control Hamiltonians the fixed points of these systems can be shifted. We have further demonstrated how to apply these techniques to preserve known quantum states for arbitrarily long times without the requirements of redundancy or error thresholds from quantum error correction.

A framework for labeling and exploring the space of stabilizable states in higher dimensional systems remains to be determined. Already for two-spins there is no intuitive description of the manifold of stabilizable states and we have only demonstrated one example of a state which can be stabilized. Combining this higher dimensional labeling with a prescription for using the stabilizable states in a coherent fashion for quantum information processing could provide a new method for protecting quantum computers from thermodynamic errors.

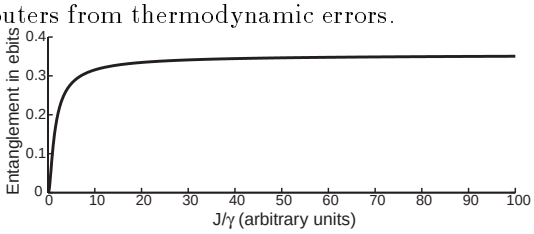


FIG. 2: The entanglement of formation of the fixed point under the Hamiltonian of equation 23.

## Acknowledgments

This work was supported in part by the Center for Bits and Atoms (NSF CCR-0122419) and the HP-MIT Alliance. We gratefully acknowledge Matthias Steffen, David Cory, and Aram Harrow for helpful comments and suggestions.

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